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Onset of thermal instability in a horizontal layer of fluid with modulated boundary temperatures

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Abstract Thermal instability in a horizontal layer of fluid, with the boundary temperatures modulated sinusoidally in time, is studied. The amplitude of modulation is assumed small and is used as an expansion parameter. It is shown that an exact solution can be obtained, even when the boundaries are considered to be rigid. When only the lower boundary temperature is modulated, for small values of the Prandtl number modulation is always stabilizing, while for large values it can be stabilizing or destabilizing depending on the modulation frequency. When both boundary temperatures are modulated in phase, modulation is destabilizing for low modulation frequency, but for higher modulation frequency stabilization occurs for low values of the Prandtl number. When the two boundary temperatures are modulated out of phase the modulation always has a stabilizing effect.

Keywords Bénard · Convection · Modulation · Thermal instability

1 Introduction

Theoretical calculations as well as experiments have been carried out to study the effect of parametric modulation on the onset of thermal instability in the Rayleigh–Bénard problem. This was inspired by the classical inverted pendulum [1,2] and the aim was to explore whether modulation can have a stabilizing effect on hydrodynamic instabilities. In the Rayleigh–Bénard problem a layer of fluid is confined between two infinite horizontal planes and it is known that instability sets in, as convection cells, when the Rayleigh number, which is a non-dimensional measure of the temperature difference across the layer, exceeds a critical value. In one of the earliest theoretical studies of the modulated Rayleigh– Bénard problem [3] the boundary temperatures were considered to be modulated sinusoidally in time. A linear stability analysis was carried out. The amplitude of modulation was assumed small and was used as an expansion parameter. For simplicity both boundaries were assumed to be stress-free. Subsequently [4,5] the more realistic situation with two rigid boundaries was tackled using expansions in terms of trial functions. In a more recent study [6] approximate solutions were obtained when the boundaries are rigid. This study again used the amplitude of oscillation as an expansion parameter and obtained closed-form solutions by using a one-term Galerkin approximation. Most of these studies considered modulation of the lower boundary temperature only. Further it has been shown [7] that when both boundary temperatures are modulated in phase the

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effect of modulation is usually destabilizing. For the unmodulated Rayleigh–Bénard problem, although assuming stress-free boundaries makes the analysis simpler [8], an exact solution is known to exist even when the boundaries are considered to be rigid [9]. In the presence of temperature modulation, if the amplitude of modulation is assumed small and used as an expansion parameter, to lowest order, the equations are same as for the unmodulated problem for which the exact solution of Pellew and Southwell can be used. The equations at higher orders too can be solved in closed form. Here we carry out such an analysis and compute the effect of boundary temperature modulation on the value of the critical Rayleigh number for onset of thermal instability and compare with previously published results.

A modulated system is a special case of an unsteady system. Unlike steady systems, for an unsteady system the normal-mode analysis is not valid and even the meaning of stability is not clear [10]. However, when the basic state, although unsteady, is periodic in time, stability can be clearly defined and in place of the normal mode analysis, the Floquet theory can be used.

2 Formulation

We consider a horizontal layer of fluid between two infinite parallel plates. Assuming the Boussinesq approximation, the governing equations for the fluid, in non-dimensional form, are

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\frac{1}{\mathrm{Pr}}\nabla p+\mathrm{Ra}T\mathbf{k}+\nabla^{2}\mathbf{v}, \tag{1}$$

$$\nabla\cdot\mathbf{v}=0, \tag{2}$$

$$\frac{\partial T}{\partial t}+\mathbf{v}\cdot\nabla T=\nabla^{2}T, \tag{3}$$

where $\mathbf{v}$ is the fluid velocity, T and p are the temperature and fluid pressure measured from a reference temperature T_0 and from the hydrostatic pressure obtained for uniform temperature T_0; ν is the kinematic viscosity, κ is the thermal diffusivity, α is the coefficient of thermal expansion, ρ_0 is the fluid density at the reference temperature T_0 and g is the acceleration due to gravity. We have assumed a Cartesian coordinate system (x, y, z) and $\mathbf{k}$ is the unit vector along z. Gravity is assumed to be in the negative z-direction. We assume a layer depth d, and a temperature difference ΔT between the lower and upper boundaries. Here coordinates, time, velocity, pressure and temperature have been scaled with reference to $d, d^2/\kappa, \kappa/d, \rho_0\kappa^2/d^2$ and ΔT. The Prandtl number Pr and the Rayleigh number Ra are defined by

$$\mathrm{Pr}=\frac{\nu}{k},\quad \mathrm{Ra}=\frac{g\alpha\Delta T d^{3}}{\nu\kappa}. \tag{4}$$

When both boundaries are assumed to be rigid and at rest, the no-slip boundary condition requires

$$\mathbf{v}=0 \quad \text{at } z=\pm 1/2. \tag{5}$$

We assume that the temperature boundary conditions imposed are

$$\begin{aligned} T &= 1+\epsilon\cos\sigma t \quad \text{at } z=-1/2,\\ T &= C\epsilon\cos\sigma t \quad \text{at } z=1/2, \end{aligned} \tag{6}$$

where ϵ and σ are the amplitude and frequency of modulation. For $C = 0$ only the lower boundary temperature is modulated while for $C = 1$ and -1 the two boundary temperatures are modulated in phase and in anti-phase.

A hydrostatic solution exists for the governing equations with boundary conditions described above. The solution is given by $\mathbf{v} = 0$, $p = \bar{p}(z, t)$ and $T = \bar{T}(z, t)$. It can be shown that [3]

$$\bar{T}=T_s(z)+\frac{\epsilon}{2}\left[T_1(z)\mathrm{e}^{\mathrm{i}\sigma t}+\widetilde{T}_1(z)\mathrm{e}^{-\mathrm{i}\sigma t}\right], \tag{7}$$

where the tilde is used to denote the complex conjugate, and

$$T_s = -\left(z - \frac{1}{2}\right), \tag{8}$$

$$T_1 = \frac{(C-1)}{2\sinh\frac{\lambda}{2}}\sinh\lambda z + \frac{(C+1)}{2\cosh\dfrac{\lambda}{2}}\cosh\lambda z, \tag{9}$$

$$\lambda = (1+\mathrm{i})\left(\frac{\sigma}{2}\right)^{1/2}. \tag{10}$$

We now consider a small perturbation from this hydrostatic solution defined by

$$\mathbf{v} = (u, v, w), \quad p = \bar{p} + \varpi, \quad T = \bar{T} + \theta. \tag{11}$$

Substituting in Eqs. 1–3, 5 and 6 and linearizing in the perturbations, we obtain

$$\frac{1}{\mathrm{Pr}}\frac{\partial u}{\partial t} = -\frac{1}{\mathrm{Pr}}\frac{\partial \varpi}{\partial x} + \nabla^2 u, \tag{12}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial v}{\partial t} = -\frac{1}{\mathrm{Pr}}\frac{\partial \varpi}{\partial y} + \nabla^2 v, \tag{13}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial w}{\partial t} = -\frac{1}{\mathrm{Pr}}\frac{\partial \varpi}{\partial z} + \mathrm{Ra}\theta + \nabla^2 w, \tag{14}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{15}$$

$$\frac{\partial \theta}{\partial t} + w\frac{\partial \bar{T}}{\partial z} = \nabla^2\theta, \tag{16}$$

$$u = v = w = \theta = 0 \quad \text{at } z = \pm 1/2. \tag{17}$$

Eliminating u, v and ϖ between Eqs. 12–15 we obtain

$$\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}\nabla^2 w = \mathrm{Ra}\nabla_1^2\theta + \nabla^4 w, \tag{18}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Using Eq. 15, boundary condition (17) can be written equivalently as

$$w = \frac{\partial w}{\partial z} = \theta = 0 \quad \text{at } z = \pm 1/2. \tag{19}$$

Linear stability is governed by Eqs. 16 and 18, with boundary condition (19), which are to be solved for w and θ. The hydrostatic solution enters through $\partial\bar{T}/\partial z$ in Eq. 16. Using Eqs. 7–9 we have

$$\frac{\partial \bar{T}}{\partial z} = -1 - \frac{\epsilon}{2}\left[f(z)\mathrm{e}^{\mathrm{i}\sigma t} + \tilde{f}(z)\mathrm{e}^{-\mathrm{i}\sigma t}\right], \tag{20}$$

where

$$f(z) = (1-C)\frac{\lambda\cosh\lambda z}{2\sinh\dfrac{\lambda}{2}} - (1+C)\frac{\lambda\sinh\lambda z}{2\cosh\frac{\lambda}{2}}. \tag{21}$$

We consider normal modes with (x, y)-dependence of the form $\exp[\mathrm{i}(a_x x + a_y y)]$ and define the wavenumber a in the horizontal plane by $a^2 = a_x^2 + a_y^2$. Further we define

$$F = a^2\mathrm{Ra}\theta. \tag{22}$$

Equations 18, 16 and 19 can now be written as

$$\left(\frac{\partial^2}{\partial z^2} - a^2 - \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial z^2} - a^2\right) w = F, \tag{23}$$

$$\left(\frac{\partial^2}{\partial z^2} - a^2 - \frac{\partial}{\partial t}\right) F = a^2\mathrm{Ra}\frac{\partial \bar{T}}{\partial z} w, \tag{24}$$

$$w = \frac{\partial w}{\partial z} = F = 0 \quad \text{at } z = \pm 1/2. \tag{25}$$

To obtain the stability boundary we assume that $\epsilon \ll 1$ and follow a procedure which has been used for the modulated Taylor–Couette problem [11,12]. We assume that on the stability boundary the perturbations are periodic in time with the same time period as the boundary temperature modulation or the hydrostatic solution. Then we can assume the expansions

$$
\begin{aligned}
w(z,t) &= w_s(z) + \sum_{n=1}^{\infty} \tfrac{1}{2}\left[w_n(z)\mathrm{e}^{\mathrm{i}n\sigma t} + \tilde{w}_n(z)\mathrm{e}^{-\mathrm{i}n\sigma t}\right],\\
F(z,t) &= F_s(z) + \sum_{n=1}^{\infty} \tfrac{1}{2}\left[F_n(z)\mathrm{e}^{\mathrm{i}n\sigma t} + \tilde{F}_n(z)\mathrm{e}^{-\mathrm{i}n\sigma t}\right].
\end{aligned} \tag{26}
$$

Further we use ϵ as an expansion parameter and assume the expansions

$$
\begin{aligned}
w_s &= w_s^{(0)} + \epsilon^2 w_s^{(2)} + \cdots,\\
F_s &= F_s^{(0)} + \epsilon^2 F_s^{(2)} + \cdots,\\
\mathrm{Ra}_c &= \mathrm{Ra}_c^{(0)} + \epsilon^2 \mathrm{Ra}_c^{(2)} + \cdots,\\
w_1 &= \epsilon w_1^{(1)} + \cdots,\\
F_1 &= \epsilon F_1^{(1)} + \cdots,
\end{aligned} \tag{27}
$$

where Ra_c is the value of Ra on the stability boundary. Substituting from Eqs. 26 and 27 in Eqs. 23–25 and separating to various orders in ϵ, we obtain to order ϵ^0

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2\right)^2 w_s^{(0)} = F_s^{(0)}, \tag{28}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2\right) F_s^{(0)} = -a^2 \mathrm{Ra}_c^{(0)} w_s^{(0)}, \tag{29}$$

$$w_s^{(0)} = \frac{\mathrm{d}w_s^{(0)}}{\mathrm{d}z} = F_s^{(0)} = 0 \quad \text{at } z = \pm 1/2. \tag{30}$$

To order ϵ^1 we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2 - \frac{\mathrm{i}\sigma}{\mathrm{Pr}}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2\right) w_1^{(1)} = F_1^{(1)}, \tag{31}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2 - \mathrm{i}\sigma\right) F_1^{(1)} = -a^2 \mathrm{Ra}_c^{(0)} w_1^{(1)} - a^2 \mathrm{Ra}_c^{(0)} f w_s^{(0)}, \tag{32}$$

$$w_1^{(1)} = \frac{\mathrm{d}w_1^{(1)}}{\mathrm{d}z} = F_1^{(1)} = 0 \quad \text{at } z = \pm 1/2. \tag{33}$$

Collecting the terms of order ϵ^2 which are independent of t, we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2\right)^2 w_s^{(2)} = F_s^{(2)}, \tag{34}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2\right) F_s^{(2)} = -a^2 \mathrm{Ra}_c^{(0)} w_s^{(2)} - \frac{1}{4} a^2 \mathrm{Ra}_c^{(0)} \left[f \tilde{w}_1^{(1)} + \tilde{f} w_1^{(1)}\right] - a^2 \mathrm{Ra}_c^{(2)} w_s^{(0)}, \tag{35}$$

$$w_s^{(2)} = \frac{\mathrm{d}w_s^{(2)}}{\mathrm{d}z} = F_s^{(2)} = 0 \quad \text{at } z = \pm 1/2. \tag{36}$$

3 Solution

The solution for the order ϵ^0 problem is known [9]. Substituting for $F_s^{(0)}$ from Eq. 28 in Eqs. 29 and 30, we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-a^2\right)^3 w_s^{(0)} = -a^2\mathrm{Ra}_c^{(0)} w_s^{(0)}, \tag{37}$$

$$w_s^{(0)} = \frac{\mathrm{d}w_s^{(0)}}{\mathrm{d}z} = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-a^2\right)^2 w_s^{(0)} = 0 \quad \text{at } z=\pm 1/2. \tag{38}$$

Defining τ by

$$\tau^3 a^4 = \mathrm{Ra}_c^{(0)}, \tag{39}$$

the solution of Eq. 37 has the form e^{qz} where q satisfies

$$(q^2-a^2)^3 = -\tau^3 a^6. \tag{40}$$

The six roots for q are given by

$$q = \pm \mathrm{i}q_0, \quad \pm q, \quad \pm\tilde{q}, \tag{41}$$

where

$$\begin{aligned} q_0 &= a(\tau-1)^{1/2}, \quad q = q_1 + \mathrm{i}q_2, \\ q_1 &= a\left\{\frac{1}{2}\left(1+\tau+\tau^2\right)^{1/2} + \frac{1}{2}\left(1+\frac{1}{2}\tau\right)\right\}^{1/2}, \\ q_2 &= a\left\{\frac{1}{2}\left(1+\tau+\tau^2\right)^{1/2} - \frac{1}{2}\left(1+\frac{1}{2}\tau\right)\right\}^{1/2}. \end{aligned} \tag{42}$$

It follows from the evenness of the operators and the identity of the boundary conditions at $z = \pm 1/2$ that the solutions of Eqs. 37 and 38 fall into two non-combining groups of even and odd solutions [13]. It is known that, as the Rayleigh number is increased, instability first sets in for an even solution. Since we are interested in finding the lowest value of the Rayleigh number for which instability sets in, we assume

$$w_s^{(0)} = \cos q_0 z + A\cosh qz + \tilde{A}\cosh\tilde{q}z. \tag{43}$$

Substituting for $w_s^{(0)}$ from Eq. 43 in boundary conditions (38), after some fairly lengthy algebra [9,13] one obtains

$$\frac{(q_1+q_2\sqrt{3})\sinh q_1 + (q_1\sqrt{3}-q_2)\sin q_2}{\cosh q_1 + \cos q_2} = -q_0\tan\frac{1}{2}q_0, \tag{44}$$

which can be solved for $\mathrm{Ra}_c^{(0)}$ as a function of a. We find that for the unmodulated configuration instability first sets in at $\mathrm{Ra}_c^{(0)} = 1707.7617$ for $a = 3.1163$. It can be readily shown that

$$A = \left(\frac{-1+\mathrm{i}\sqrt{3}}{2}\right)\frac{\cos\frac{1}{2}q_0}{\cosh\frac{1}{2}q}. \tag{45}$$

Further

$$F_s^{(0)} = \tau^2 a^4[\cos q_0 z + A_F\cosh qz + \tilde{A}_F\cosh\tilde{q}z], \tag{46}$$

where

$$A_F = \left(\frac{-1-\mathrm{i}\sqrt{3}}{2}\right)\frac{\cos\frac{1}{2}q_0}{\cosh\frac{1}{2}q}. \tag{47}$$

For the order ϵ^1 system, Eqs. 31–33, again substituting for $F_1^{(1)}$ from Eq. 31 in Eqs. 32 and 33, we obtain

$$\left[\left(\frac{d^2}{dz^2}-a^2-i\sigma\right)\left(\frac{d^2}{dz^2}-a^2-\frac{i\sigma}{Pr}\right)\left(\frac{d^2}{dz^2}-a^2\right)+a^2 Ra_c^{(0)}\right]w_1^{(1)}=-a^2 Ra_c^{(0)} f w_s^{(0)}, \tag{48}$$

$$w_1^{(1)}=\frac{dw_1^{(1)}}{dz}=\left(\frac{d^2}{dz^2}-a^2-\frac{i\sigma}{Pr}\right)\left(\frac{d^2}{dz^2}-a^2\right)w_1^{(1)}=0 \quad \text{at } z=\pm 1/2. \tag{49}$$

Using Eq. 21 and 43

$$f w_s^{(0)}=\left[(1-C)\frac{\lambda\cosh\lambda z}{2\sinh\frac{\lambda}{2}}-(1+C)\frac{\lambda\sinh\lambda z}{2\cosh\frac{\lambda}{2}}\right]\times\left(\cosh iq_0 z+A\cosh qz+\tilde{A}\cosh\tilde{q}z\right). \tag{50}$$

Substituting in Eq. 48 we can write the solution for $w_1^{(1)}$ as the sum of a particular integral and a complementary function

$$w_1^{(1)}=w_{1,pi}^{(1)}+w_{1,cf}^{(1)}. \tag{51}$$

A particular integral is given by

$$\begin{aligned}
w_{1,pi}^{(1)}=&-a^2 Ra_c^{(0)}\left(\frac{1-C}{2}\right)\frac{\lambda}{2\sinh\frac{\lambda}{2}}\left\{\left[\frac{\cosh(\lambda z+iq_0 z)}{M(\lambda+iq_0)}+\frac{\cosh(\lambda z-iq_0 z)}{M(\lambda-iq_0)}\right]\right.\\
&\left.+A\left[\frac{\cosh(\lambda z+qz)}{M(\lambda+q)}+\frac{\cosh(\lambda z-qz)}{M(\lambda-q)}\right]+\tilde{A}\left[\frac{\cosh(\lambda z+\tilde{q}z)}{M(\lambda+\tilde{q})}+\frac{\cosh(\lambda z-\tilde{q}z)}{M(\lambda-\tilde{q})}\right]\right\}\\
&+a^2 Ra_c^{(0)}\left(\frac{1+C}{2}\right)\frac{\lambda}{2\cosh\frac{\lambda}{2}}\left\{\left[\frac{\sinh(\lambda z+iq_0 z)}{M(\lambda+iq_0)}+\frac{\sinh(\lambda z-iq_0 z)}{M(\lambda-iq_0)}\right]\right.\\
&\left.+A\left[\frac{\sinh(\lambda z+qz)}{M(\lambda+q)}+\frac{\sinh(\lambda z-qz)}{M(\lambda-q)}\right]+\tilde{A}\left[\frac{\sinh(\lambda z+\tilde{q}z)}{M(\lambda+\tilde{q})}+\frac{\sinh(\lambda z-\tilde{q}z)}{M(\lambda-\tilde{q})}\right]\right\},
\end{aligned} \tag{52}$$

where

$$M(l)=\left(l^2-a^2-i\sigma\right)\left(l^2-a^2-\frac{i\sigma}{Pr}\right)\left(l^2-a^2\right)+a^2 Ra_c^{(0)}. \tag{53}$$

The complementary function can be written as

$$w_{1,cf}^{(1)}=B_1\cosh r_1 z+B_2\cosh r_2 z+B_3\cosh r_3 z+B_4\sinh r_1 z+B_5\sinh r_2 z+B_6\sinh r_3 z, \tag{54}$$

where $r=r_1, r_2, r_3$ are the solutions of

$$\left(r^2-a^2-i\sigma\right)\left(r^2-a^2-\frac{i\sigma}{Pr}\right)\left(r^2-a^2\right)+a^2 Ra_c^{(0)}=0. \tag{55}$$

Substituting for $w_1^{(1)}$ from Eqs. 51, 52 and 54 in boundary conditions (49) we obtain six equations which can be solved for the six coefficients $B_1,\ldots,B_6$. Thus the first-order solution for $w_1^{(1)}$ is known.

For the order ϵ^2 system, Eqs. 34–36, the solvability condition requires

$$Ra_c^{(2)}=-\frac{1}{4}Ra_c^{(0)}\frac{\int_{-1/2}^{1/2}\bar{F}_s^{(0)}\left[f\tilde{w}_1^{(1)}+\tilde{f}w_1^{(1)}\right]dz}{\int_{-1/2}^{1/2}\bar{F}_s^{(0)}w_s^{(0)}dz}, \tag{56}$$

where $\bar{F}_s^{(0)}$ is the solution of the adjoint problem

$$\left(\frac{d^2}{dz^2}-a^2\right)^2\bar{w}_s^{(0)}+a^2 Ra_c^{(0)}\bar{F}_s^{(0)}=0, \tag{57}$$

$$-\bar{w}_s^{(0)}+\left(\frac{d^2}{dz^2}-a^2\right)\bar{F}_s^{(0)}=0, \tag{58}$$

$$\bar{w}_s^{(0)}=\frac{d\bar{w}_s^{(0)}}{dz}=\bar{F}_s^{(0)}=0, \quad z=\pm 1/2. \tag{59}$$

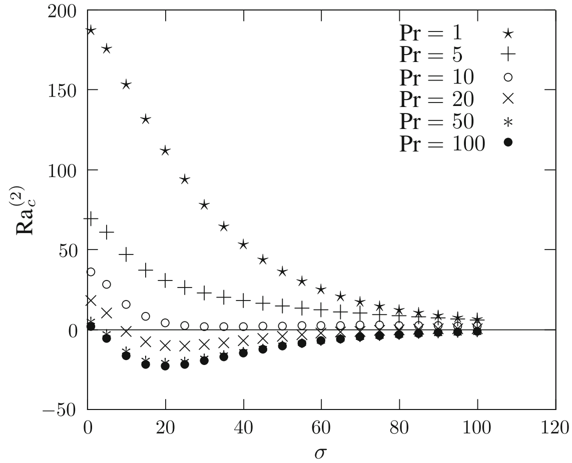

Fig. 1 $\mathrm{Ra}_c^{(2)}$ versus σ for different values of Pr for $C = 0$, i.e., only the lower boundary temperature is modulated

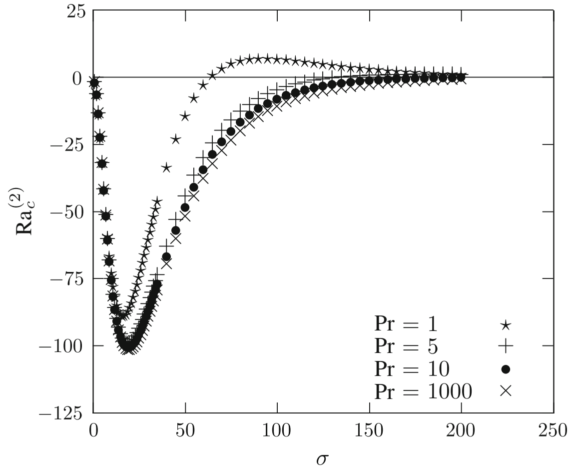

Fig. 2 $\mathrm{Ra}_c^{(2)}$ versus σ for different values of Pr for $C = 1$, i.e., both boundary temperatures are modulated in phase

By rescaling $\bar{w}_s^{(0)} \to -a^2\mathrm{Ra}_c^{(0)}\bar{w}_s^{(0)}$ it is easily seen that the adjoint equations become identical to the zeroth order system given by Eqs. 28–30. Therefore, with this rescaling the solution of the zeroth order system is also the solution of the adjoint problem. Further since both numerator and denominator in the expression for $\mathrm{Ra}_c^{(2)}$ are linear in $\bar{F}_s^{(0)}$, multiplying $\bar{F}_s^{(0)}$ by a constant does not change the value of $\mathrm{Ra}_c^{(2)}$. Therefore, using (46) we can write

$$\bar{F}_s^{(0)} = \cos q_0 z + A_F \cosh qz + \tilde{A}_F \cosh \tilde{q}z. \tag{60}$$

With $w_1^{(1)}$ and $\bar{F}_s^{(0)}$ known, we can readily compute $\mathrm{Ra}_c^{(2)}$ using (56).

4 Results and discussion

Using Eq. 56 we have computed $\mathrm{Ra}_c^{(2)}$ for various values of Pr and σ for $C = 0, 1, -1$. Figure 1 shows $\mathrm{Ra}_c^{(2)}$ as a function of σ for various values of Pr for $C = 0$, i.e., only the lower boundary temperature is modulated. For $\mathrm{Pr} = 1, 5$, and 10, $\mathrm{Ra}_c^{(2)}$ is positive, consequently in the presence of modulation instability sets in at a higher value of the Rayleigh number and, therefore, modulation has a stabilizing effect. For $\mathrm{Pr} = 20$, for low values of σ modulation is stabilizing but for higher values of σ, $\mathrm{Ra}_c^{(2)}$ is negative and, therefore modulation is destabilizing. For $\mathrm{Pr} = 50$ and 100 the stabilizing region decreases and in the destabilizing region the destabilizing effect is larger. For $C = 1$, i.e., both boundary temperatures modulated in phase, Fig. 2 shows $\mathrm{Ra}_c^{(2)}$ as a function of σ for various values of Pr. We observe that there is always a range of σ for which modulation has a destabilizing effect. For $C = -1$, i.e., when the two boundary temperatures are modulated in anti-phase, Fig. 3 shows $\mathrm{Ra}_c^{(2)}$ as a function of σ for various values of Pr. Here modulation always has a stabilizing effect. These results are qualitatively similar to the results in [3] obtained assuming both boundaries to be stress-free. Our results are also in agreement with results in [4] and [6] which show that for $C = 0$ and low values of Pr, modulation has a stabilizing effect. Our results showing destabilization for $C = 1$, the two boundary temperatures modulated in phase, are in agreement with [7].

We now make a quantitative comparison with previously published results. In Table 1 we show $\mathrm{Ra}_c^{(2)}$ computed using (56) together with the results obtained using the method in [6], for $C = 0$. We observe that for $\sigma = 1$ the approximate and exact results differ typically by 1% and never by more than 5%. However, for higher values of σ the two can differ by as much as 15%. In [6] the results were compared with those in [4]. Neither of these is an exact solution as can be seen from their zeroth order results. While [4] predicts onset of instability for $\mathrm{Ra} = 1728$

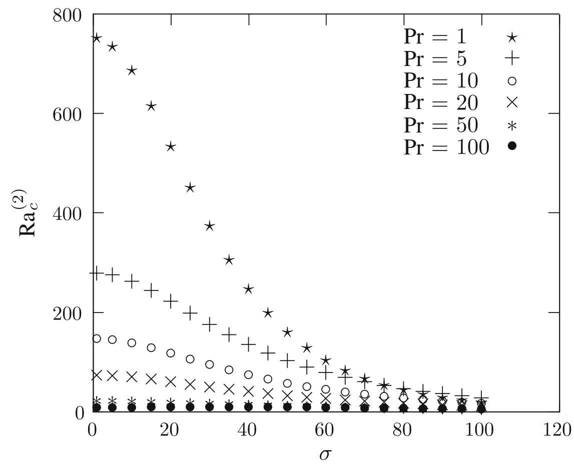

Fig. 3 $\mathrm{Ra}_c^{(2)}$ versus σ for different values of Pr for $C = -1$, i.e., the two boundary temperatures are modulated in anti-phase

Table 1 Comparison of $\mathrm{Ra}_c^{(2)}$ computed using the exact solution of the present study and the approximate solution in [6] for $C = 0$

σ	Pr	$\mathrm{Ra}_c^{(2)}$	
		Ref. [6]	Present study
1	1	185.89	188.42
1	2	134.97	136.15
1	5	70.14	69.46
1	10	38.58	36.95
5	1	181.64	176.78
10	1	169.29	154.43
20	1	130.81	113.08
30	1	90.80	79.35
40	1	59.72	54.49
50	1	38.61	37.50

and $a = 3.10$, [6] and [13] give the values as Ra = 1715.08 and $a = 3.117$. The exact solution gives Ra = 1707.76 and $a = 3.116$. Our results showing destabilization when $C = 1$ are also in fairly good quantitative agreement with the results in [7]. As a further check on our analytical results we have also computed $\mathrm{Ra}_c^{(2)}$ by carrying out a finite-difference solution of (28–33), together with numerical evaluation of the integrals in (56). This closely follows the method in [12]. Using this method, with 99 internal grid-points and assuming $a = 3.1163$, for $\sigma = 1$ and Pr = 1 we obtain $\mathrm{Ra}_c^{(2)} = 188.29$, which is in good agreement with our analytical results. Similarly for $\sigma = 1$ and Pr = 10 we obtain $\mathrm{Ra}_c^{(2)} = 36.98$ and for $\sigma = 10$ and Pr = 1 we obtain $\mathrm{Ra}_c^{(2)} = 154.28$, again in close agreement with our analytical results. We have not found exact agreement with any of the previously published results since all these earlier studies used truncated expansions and consequently their results were not exact. On the other hand, our results we believe are exact. Therefore, these we hope can serve as benchmark values.

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